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On the validity of the paraxial eikonal in catastrophe optics

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Abstract. The paraxial eikonal $\Psi = Z - f(\boldsymbol{r}) + (\boldsymbol{R} - \boldsymbol{r})^2/2Z$ is frequently used in catastrophe optics. We show that, except in the far-field limit, it can lead not only to quantitative errors, but to major qualitative errors unless the singularity involved is three-determinate. If at some point on the local optical axis ($\boldsymbol{R} = 0$) the exact eikonal has a cuspoid singularity or an umbilic singularity with non-zero four-jet, then the paraxial eikonal generically has a four-determinate singularity at that point. The consequences for cusp and swallowtail foci are explored in detail. We show that such errors do not result from any obvious failure to satisfy the conditions under which Ψ is derived.

1. Introduction

Paraxial optics deals with bundles of rays at small angles to each other (Born and Wolf, p 193). At a focus, neighbouring rays coalesce, so a sufficiently small neighbourhood of a (typical) focus must be describable by paraxial optics. A *paraxial approximation* is any approximation valid for a paraxial ray bundle; when introduced at the earliest possible stage the result is the *paraxial eikonal* (derived later; see also Born and Wolf (1975, pp 112–3)).

The study of generic (i.e. typical) foci is the subject of catastrophe optics (Berry and Upstill 1980, and references therein; Poston and Stewart 1978), in which the paraxial eikonal has been widely used. We show here that the paraxial eikonal is incapable of giving a description, consistent with the laws of geometrical optics, of any caustic organised by an eikonal singularity which is not four-determinate, and is only completely reliable for three-determinate singularities. (Loosely: A function $f: D \to \mathbb{R}$ is *k-determinate* at some point $x \in D$ if it is *right-equivalent* to its *k-jet*, which is its Taylor polynomial of degree k about x. Two functions $f, g: D \to \mathbb{R}$ are *right-equivalent* if $f(x) = g(T(x))$ for all $x \in D$, for some $T: D \to D$ which is a *diffeomorphism*, or smooth change of coordinates. Determinacy is explained simply by Poston and Stewart (1978, ch 8) and more mathematically in the review article by Zeeman (1982).) Interestingly, all caustics so far studied in any detail have four-determinate singularities. The failure of the paraxial eikonal is subtle and, as we shall show, is not caused by any obvious violation of the conditions under which it is derived.

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2. The paraxial eikonal

Optics in the ray limit may be described by the eikonal Φ, which is the optical distance of an observation point P from a general point on an equiphase surface S, called the *initial wavefront*, whose shape may be found from the geometry of the system (Berry and Upstill 1980, § 4.1, pp 297–9). S may always be chosen to be locally smooth, so that we may represent it locally as the graph $\{(x, y, z)\in\mathbb{R}^3 : z=f(x, y)\}$ of a smooth function $f:\mathbb{R}^2\to\mathbb{R}$, and we represent P by $(X, Y, Z)\in\mathbb{R}^3$. Taking the refractive index as unity for simplicity gives

$$\Phi(x, y; X, Y, Z)=[(Z-f(x, y))^2+(X-x)^2+(Y-y)^2]^{1/2}. \tag{1}$$

This formula is awkward to work with because of the square root and the $(f(x, y))^2$ term. (Although the square root can be avoided when studying purely geometrical properties by working with Φ^2, diffraction studies need Φ itself.)

If the ray reaching (X, Y, Z) from $(x, y, f(x, y))$ on S makes a small angle to the z axis then

$$(Z-f(x, y))^2 \gg (X-x)^2+(Y-y)^2. \tag{2a}$$

Taking the z axis as local optical axis gives (2a) as the paraxiality condition. Imposing also the condition that

$$Z \gg |f(x, y)| \tag{2b}$$

justifies expanding (1) in inverse powers of Z and retaining only the three leading-order terms to give the paraxial eikonal

$$\Psi(x, y; X, Y, Z)=Z-f(x, y)+[(X-x)^2+(Y-y)^2]/2Z, \tag{3}$$

so that $\Psi=\Phi+\mathrm{O}(Z^{-2})$. Note that the paraxial eikonal is defined in terms of a specific coordinate system, whereas the exact eikonal is coordinate independent. Condition (2a) will be satisfied globally if $|\nabla f(x, y)|\ll 1$ everywhere, and hence S has small curvature. Then a coordinate system may be chosen so that $|f(x, y)|$ is small everywhere, and (2b) will be satisfied for all $Z\gg 0$. In particular, (2b) will be satisfied near foci.

The paraxial eikonal (3) may be either

(a) analysed globally to determine the (approximate) caustic structure of the wavefield, or

(b) used to examine the neighbourhood of a specific point on the caustic, whose exact location is already known.

We are primarily concerned with the latter application, to a focus arising from a specific point, such as an umbilic point (Berry and Upstill 1980, § 4.3, pp 301–5), of the initial wavefront S, whose location is already known exactly from its geometrical properties. This application includes all the detailed analyses of diffraction catastrophes performed to date (e.g. A_3 and D_4^- by Berry *et al* (1979); D_5 and E_6 by Nye (1979)—reviewed by Berry and Upstill (1980)—and ${}^0X_9$ by Upstill *et al* (1982)). (We assume that S is actually a member of a family of initial wavefronts, which is sufficiently large that it is generic for the focus we seek to occur.) In this context we can ensure that conditions (2a) and (2b) are satisfied without constraining the slope or curvature of S by making a suitable choice of coordinate system, as follows. Put the origin at the point on S from which the focus originates, so that $f(0, 0)=0$, and orient the axes so that the (x, y) plane is tangent to S at the origin, i.e. $\nabla f(0, 0)=0$. Then the ray from the origin coincides with the z axis. Furthermore, by a rotation of the (x, y)

coordinates we can always put f into the form

$$f(x, y) = \tfrac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2) + \sum_{m+n \geq 3} f_{mn} x^m y^n \tag{4}$$

where κ_1 and κ_2 are the principal curvatures of S at the origin.

3. Validity of the paraxial eikonal in catastrophe optics

Within the local context of catastrophe optics only rays originating in a neighbourhood of the origin are relevant. Conditions (2a) and (2b) may be satisfied for *all P* lying on rays from this neighbourhood simply by taking it small enough, by virtue of our choice of coordinate system and the smoothness of f. Now the only role played directly by the curvature of S is to determine how small the neighbourhood must be. Hence it appears that Ψ should be a valid approximation to Φ in the neighbourhood of a generic focus.

Of course, we expect the description of a caustic by Ψ to contain metrical errors, whose magnitudes should decrease as Z increases. But can Ψ introduce qualitative errors? This is the question addressed in this paper.

We can make the *numerical* error in approximating Φ by Ψ as small as we like by restricting our attention to a sufficiently small neighbourhood of the focus and of the origin; in particular *at* the focus $\Phi = \Psi = Z$. But this is not enough. For Ψ to give a qualitatively correct description of the focus it must be *right-equivalent* to Φ (Poston and Stewart 1978). That this requirement may not be satisfied is easily illustrated in two dimensions (x, z) by the parabolic initial wavefront given by $f(x) = ax^2$. This is well known to focus onto a cusped caustic with its cusp point at $X = 0$, $Z = 1/2a$, as seen by solving $\partial\Phi^2/\partial x = \partial^2\Phi^2/\partial x^2 = 0$. The cusp results from the essential quartic nature of Φ^2; however, $\Psi = (1/2Z - a)x^2 - (X/Z)x + (Z + X^2/2Z)$ is only quadratic. It does not describe any stable focus—at best it indicates its own inadequacy by becoming indeterminate, and hence unstable, at what should be the cusp point. In fact, Ψ describes only one of the three rays involved in the cusp focus, that from smallest $|x|$.

This example is extreme, but clearly shows that Ψ can indeed lead to immense qualitative errors. A complete discussion would require an analysis of Φ and Ψ in the full parameter space, that is, a comparison of the catastrophes or unfoldings that they generate. However, we shall restrict ourselves here to answering the simpler preliminary question: if Φ has a particular singularity at some focus on the local optical axis (the Z axis), what singularities can Ψ display at that focus?

4. The axial singularities

Consider a focus at $P_0 \equiv (0, 0, Z_0)$; the focusing condition requires that P_0 lies on a curvature centre of S, so we assume throughout that $Z_0 = 1/\kappa_1$. Then $\Phi(x, y; P)$ and $\Psi(x, y; P)$ with variable $P \equiv (X, Y, Z)$ unfold the singularities $\varphi(x, y) \equiv \Phi(x, y; P_0)$ and $\psi(x, y) \equiv \Psi(x, y; P_0)$ respectively. The true catastrophe type of the focus at P_0 is determined by the singularity φ exhibited by the exact eikonal at the origin, which depends on its Taylor coefficients

$$\varphi_{mn} \equiv (1/m!\,n!)(\partial^{m+n}\varphi/\partial x^m\,\partial y^n)(0, 0).$$

To determine whether ψ is right-equivalent to φ we must compare their Taylor coefficients, defining ψ_{mn} analogously to φ_{mn}.

From (1), (3) and (4) it is easily computed that:

for $m+n=2$:

$$\varphi_{20}=\psi_{20}=0, \qquad \varphi_{11}=\psi_{11}=0, \qquad \varphi_{02}=\psi_{02}=\tfrac{1}{2}D,$$

where $D\equiv\kappa_1-\kappa_2$ is the *curvature difference.* At an umbilic point $\kappa_1=\kappa_2$ (by definition) so that $D=0$.

for $m+n=3$:

$$\varphi_{mn}=\psi_{mn}=-f_{mn};$$

for $m+n=4$:

$$\varphi_{40}=-f_{40}+\tfrac{1}{8}\kappa_1^3, \qquad \varphi_{31}=-f_{31}, \qquad \varphi_{22}=-f_{22}+\tfrac{1}{4}\kappa_1^2\kappa_2,$$

$$\varphi_{13}=-f_{13}, \qquad \varphi_{04}=-f_{04}+\tfrac{1}{8}\kappa_1(\kappa_2^2-D^2);$$

whereas $\psi_{mn}=-f_{mn}$;

and for $m+n\geqslant 5$, we find for example that:

$$\varphi_{50}=-f_{50}+\tfrac{1}{2}\kappa_1^2 f_{30}, \qquad \varphi_{05}=-f_{05}+\tfrac{1}{2}\kappa_1^2 f_{03},$$

$$\varphi_{60}=-f_{60}+\tfrac{1}{2}\kappa_1^2 f_{40}, \qquad \varphi_{06}=-f_{06}+\tfrac{1}{2}\kappa_1^2\{f_{04}-\tfrac{1}{8}D(\kappa_2^2-D^2)\},$$

whereas $\psi_{mn}=-f_{mn}$ (for all $m+n\geqslant 3$, in fact).

Only for $\kappa_1=0$ do all Taylor coefficients of φ and ψ agree, which occurs only for far-field (limit $Z_0\to\infty$) foci because $Z_0=1/\kappa_1$. We exclude this relatively trivial case. In general, Taylor coefficients of φ and ψ are equal only up to third order. Therefore only for three-determinate singularities—the fold and elliptic and hyperbolic umbilics—can the paraxial eikonal be guaranteed always to give a correct local description; otherwise further analysis of its validity is required. This is consistent with the observation (Berry and Upstill 1980, equations (4.3) and (4.4)) that the paraxial eikonal correctly describes isolated paraxial rays and their simplest focusing condition.

5. Cusp and swallowtail foci

Foci generic in three-space, and hence singularities with codimension $\leqslant 3$, are particularly important, so let us consider the cusp and swallowtail in more detail. Assuming a cuspoid singularity, the splitting lemma (Poston and Stewart 1978) allows us to write the eikonal as a Morse form in one variable plus a residual singularity in the other. Defining the nth degree coefficients of the residual singularities in φ and ψ to be C_{ne} and C_{np} respectively, we find that

$$C_{1e}=C_{1p}=C_{2e}=C_{2p}=0,$$

$$C_{3e}=\varphi_{30}=-f_{30}, \qquad C_{3p}=\psi_{30}=-f_{30},$$

$$C_{4e}=\varphi_{40}-\varphi_{21}^2/2D=-f_{40}+\tfrac{1}{8}\kappa_1^3-f_{21}^2/2D, \tag{5a}$$

$$C_{4p}=\psi_{40}-\psi_{21}^2/2D=-f_{40}-f_{21}^2/2D, \tag{5b}$$

that is:

$$C_{3\mathrm{e}} = C_{3\mathrm{p}} = -f_{30} \qquad \text{but} \qquad C_{4\mathrm{e}} = C_{4\mathrm{p}} + \tfrac{1}{8}\kappa_1^3. \tag{5c}$$

Generally $C_{n\mathrm{e,p}}$ depend on derivatives of the eikonals of order up to n, so that $C_{n\mathrm{e}} \neq C_{n\mathrm{p}}$ for $n \geqslant 4$.

The exact eikonal singularity φ is a cusp (dual cusp) if and only if

$$f_{30} = 0 \qquad \text{and} \qquad C_{4\mathrm{e}} > 0 \qquad (C_{4\mathrm{e}} < 0), \tag{6}$$

and a necessary condition for it to be a swallowtail is that

$$f_{30} = C_{4\mathrm{e}} = 0;$$

the conditions on ψ are similar. Therefore φ and ψ are both right-equivalent to the same cusp singularity if and only if $C_{4\mathrm{e}}$ and $C_{4\mathrm{p}}$ are both non-zero and have the same sign. This is so if the cusp focus is sufficiently close to a 'generic umbilic' ($\mathrm{D}_4^{\pm}$) focus, since

$$C_{4\mathrm{p}} \sim C_{4\mathrm{e}} \sim -f_{21}^2/2D \to \infty$$

as the umbilic focus is approached (because $\kappa_1 = \kappa_2$ implies that $D = 0$, and it can be shown that f_{21} is non-zero) consistent with our previous observation that generic umbilic foci are correctly described by the paraxial eikonal. Generally, from (5c) and assuming $\kappa_1 > 0$, φ is a standard cusp if ψ is, and ψ is a dual cusp if φ is, but no other correspondences necessarily hold, so that ψ reliably predicts only *standard* cusps.

If, however, φ is a swallowtail singularity ($C_{4\mathrm{e}} = 0$), then ψ can be a swallowtail ($C_{4\mathrm{p}} = 0$) only if $\kappa_1 = 0$, that is, in the far field (but then a swallowtail is no longer generic!). Consequently, a paraxial analysis to find a swallowtail focus along the correct ray will fail, and could make it appear impossible to produce a swallowtail focus, despite it being generic.

6. The general axial focus

Similar arguments apply to higher singularities that may generically occur if the system depends on additional control parameters. Our results may be summarised in terms of *paraxial determinacy*—we define a focus to be:

paraxially determined if φ and ψ are always right-equivalent (paraxial determinacy is equivalent to three-determinacy);

paraxially semidetermined if φ and ψ may be right-equivalent;

paraxially undetermined if φ and ψ cannot be right-equivalent (except in the far field).

In table 1 we list all possible exact-eikonal singularities, their k-determinacies and the possible singularities in the corresponding paraxial eikonal. Note that exchanging the roles of the exact and paraxial singularities in the table yields the possible singularities of the exact eikonal corresponding to a given paraxial singularity (up to duality). Table 1 was obtained by considering the degeneracy and non-degeneracy conditions (the generalisation of conditions (6) for the cusp above) that determine the

Table 1. Paraxial singularities resulting from specific exact singularities. Notation is from Arnol'd (1975), with the addition of ± signs to distinguish singularities that are distinct for real variables (except that for modal singularities we have not distinguished homeomorphism subclasses).

Class	Exact singularity: Common name	Arnol'd symbol	k-det.	Para. det.?	Resulting paraxial singularity
Cuspoids	Ray	A_1	2	Yes	A_1
A	Fold	A_2	3	Yes	A_2
(simple)	Cusp	$+A_3$, $-A_3$	4	Semi	$\pm A_3$, $-A_3$ (or A_k, $k \geqslant 4$)
	Higher	A_k, $k \geqslant 4$	$\geqslant 5$	No	$-A_3$
Conic umbilics	Hyperbolic Elliptic	$D_4^{\pm}$	3	Yes	$D_4^{\mp}$
D	Parabolic	D_5	4	Semi	$\pm D_5$ (or $D_k^{\pm}$, $k \geqslant 6$)
(simple)	Higher	D_k, $k \geqslant 6$	$\geqslant 5$	No	$-D_5$
Exceptional umbilics	Symbolic	E_6	4	Semi	$\pm E_6$ (or E_k, $k \geqslant 7$ or J)
E (simple) **J** (model)	Higher	E_7, E_8, **J**	4, 5, $\geqslant 6$	No	$-E_6$
Zero-three-jet umbilics	(Double cusp)	X_9	4	Semi	$\pm X_9$ (or higher X)
X and **N**	Non-zero-four-jet	X ($\neq X_9$)	$\geqslant 5$	No	$\pm X_9$ (or higher X or N)
(modal)	Zero-four-jet	N	$\geqslant 5$	No	$X \neq X_9$

type of a singularity, and investigating the effect of the change in the four-jet of φ when it is paraxially approximated by ψ.

Note that the paraxial eikonal preserves the distinction among the four classes: cuspoids (**A**); conic umbilics (**D**); exceptional umbilics (**E**, **J**); zero-three-jet umbilics (**X**, **N**), because the distinction is determined by the three-jet, which is preserved in the paraxial eikonal. The symbols for the sequences of singularities are due to Arnol'd (1975): **N** is a general name for all zero-four-jet umbilics (i.e. corank two singularities), which have not so far been further classified (Arnol'd 1975, § 13V).

The paraxial singularities in parentheses are non-generic cases resulting from special choices of initial wavefront (of which our parabolic model is an extreme example). Such special cases can occur only for paraxially-semidetermined singularities and for zero-three-jet umbilics, as shown in the table. However, it is easy to invent an f making ψ any desired singularity (although then the true singularity φ will generically be four-determinate!).

We may summarise our results by observing that if φ has any singularity other than a zero-four-jet umbilic (N) then ψ is generically four-determinate (i.e. is determined by its k-jet with $k \leqslant 4$). This is because the perturbation of φ produced by the paraxial approximation is of order four, and in the umbilic case takes the form $(x^2+y^2)^2+O(5)$, as seen from the Taylor coefficients given earlier.

One role of the initial wavefront curvature, and hence its focusing height, in determining the qualitative validity of the paraxial eikonal has now emerged: as Z_0

increases the probability increases that for paraxially semidetermined foci φ and ψ are right-equivalent, because the differences between their fourth derivatives decreases.

7. Exact and paraxial unfoldings

In the remainder of this paper we return to considerations of the full eikonals Φ and Ψ, rather than only their axial singularities φ and ψ. Recall that for Φ generically to display a singularity φ of codimension K, it must depend on at least K independent parameters, that is, $K-3$ in addition to X, Y, Z. A generic singularity is structurally stable, which means that it survives a sufficiently small perturbation—all that happens is that it is displaced slightly. Note that here *small* means (loosely speaking) small in the C^∞ norm, which is vastly more restrictive than just numerically small, i.e. small in the C^0 (or uniform) norm.

If the paraxial eikonal Ψ constituted such a small perturbation of Φ, then Ψ would be right-equivalent to Φ and one could appeal to structural stability to show that any singularity φ of Φ would still occur in Ψ, although probably displaced. But we have shown by two counter-examples involving the cusp that Ψ may not be right-equivalent to Φ for any finite Z_0. Firstly, in our example of a parabolic initial wavefront, the caustic structure of Φ was totally lost in Ψ. Secondly, our general analysis of a cusp singularity showed that for given f_{mn}, $m+n \geqslant 3$, and κ_2 there exists a non-zero range of values of κ_1 for which Φ and Ψ display cusp singularities of opposite duality at the same point on the axis, so that it seems most unlikely that Ψ and Φ can be right-equivalent. In this case Ψ would give completely erroneous phase structure for the diffraction pattern, although it might well give qualitatively correct amplitude structure, and hence a qualitatively correct caustic.

We conclude that in general Ψ and Φ may or may not be right-equivalent as unfoldings, and further investigation is required to ascertain the conditions under which they are equivalent. Clearly the condition that the focusing height Z_0 be large is neither necessary nor sufficient. What we have shown is that if Ψ and Φ are equivalent, and the main singularity is displaced, then it must be displaced off the local optical axis, because the focusing height $Z_0 = 1/\kappa_1$ on the axis is preserved by the paraxial eikonal. In fact, for a singularity with codimension >3, the displacement may well involve the 'additional' parameters in the initial wavefront S (which may be regarded as the Taylor coefficients of f) and make it disappear from the 'observation subspace' (X, Y, Z), to be recovered only by selecting a different and nearby member of the family of initial wavefronts.

Our results in table 1 show which exact-eikonal singularities will, or may, be either displaced or destroyed (we cannot at present distinguish) by the paraxial eikonal and what paraxial singularities can appear in their place. We express the survival in the paraxial eikonal of an exact singularity in terms of its paraxial determinacy. One observes that the paraxial determinacy of a singularity depends primarily upon its k-determinacy, whereas its structural stability depends upon its codimension. The distinction is that paraxial determinacy requires that not only the character, but also the position, of the singularity be unchanged by the perturbation induced by paraxial approximation. This is a much stronger stability condition than structural stability, which is stability of character only, but under general perturbations. Analysis of paraxial unfoldings would allow a determination of *paraxial stability* of foci, that is, structural stability under the perturbation $\Phi \rightarrow \Psi$.

8. Previous studies using the paraxial eikonal

As an application of our analysis, we observe that the paraxial analysis by Nye (1979) of the paraxially semidetermined foci D_5 and E_6 is valid, but similar analysis for paraxially undetermined foci, such as $D_6^{\pm}$ or E_7, would fail to find them. At first sight Nye (1978, appendix C) appears to have violated our predictions by analysing a butterfly catastrophe (A_5) using the paraxial eikonal. However, he also found the source on the initial wavefront of the butterfly foci using a small-slope approximation where the slope was non-zero, so that what he found are not the exact butterfly points. He essentially performed a (completely self-consistent) global study, our case (a) earlier, which is not the subject of our present considerations. The paraxial eikonal probably works better for paraxially semidetermined foci with high symmetry, because the resultant constraints will help preserve the equivalence between φ and ψ—see for example the paraxial analysis of ${}^0X_9$ by Upstill *et al* (1982). It is interesting that all the foci that have been studied in any detail experimentally are at least paraxially semidetermined.

9. Modal singularities, beak-to-beak and lips events

We finish with two examples of more subtle qualitative failures of the paraxial eikonal, which occur despite it giving essentially the right type of focus. Firstly, at least one unimodal singularity, X_9, is important in catastrophe optics (Berry and Upstill 1980) and ${}^0X_9$ diffraction has been studied in detail by Upstill *et al* (1982). A *modal* singularity is a continuous family, parametrised by its *moduli*, of singularities that are not right-equivalent (under diffeomorphism). Any approximation is much less likely to give the right member of a continuous family than of a discrete family, so one expects the paraxial eikonal to give the wrong moduli: indeed it does give the wrong modulus for ${}^0X_9$ (see Upstill *et al* (1982), especially appendix A). In the mid to far field this error does not appear to be important in practice, although in principle it produces a serious topological error.

Our second example involves beak-to-beak and lips events. Consider the initial wavefront S given by

$$f(x,y)=\tfrac{1}{2}(\kappa_1 x^2+\kappa_2 y^2)+ax^2y+bx^4.$$

Let $Z_0=1/\kappa_1>0$, $D\equiv\kappa_1-\kappa_2\neq 0$ and assume that $C_{4\mathrm{p}}$ $(=-b-a^2/2D)$ and $C_{4\mathrm{e}}$ $(=C_{4\mathrm{p}}+\frac{1}{8}\kappa_1^3)$ (defined as in (5*a*) and (5*b*)) are both non-zero and have the same sign. Then from (6), because $f_{30}=0$, φ and ψ are both equivalent to the same cusp singularity and the observation point P_0 lies on a rib (cusp line) of the caustic produced by S.

Using scaled variables $P'\equiv(X',Y',Z')\equiv(X,Y,Z-Z_0)/Z_0$, the tangent to the rib at $P'=0$ lies in the (Y',Z') plane and makes an angle α, given by $\tan\alpha=-2aZ_0/D$, with the Y' axis. Rotate the (Y',Z') coordinates into (Y'',Z'') so that the rib is tangent to the Y'' axis. Then Φ is locally right-equivalent to the normal form

$$C_{4\mathrm{e}}x^4+(-\kappa_1 QZ''/2D+A_{\mathrm{e}}Y''^2)x^2-X'x+\tfrac{1}{2}Dy^2$$

where $Q\equiv(D^2+4a^2Z_0^2)^{1/2}$ and $A_{\mathrm{e}}\equiv a^2/DQ^2+\kappa_2/4Z_0^2Q^2$. Similarly, Ψ is locally right-equivalent to the same normal form with $C_{4\mathrm{e}}\to C_{4\mathrm{p}}$ and $A_{\mathrm{e}}\to A_{\mathrm{p}}\equiv a^2/DQ^2$. (The second term in A_{e} results essentially from the x^2y^2 term in Φ, which is absent in Ψ.)

Let E be any plane containing the Y'' axis and transverse to the Z'' axis. From the normal form for Φ we infer that the intersection of the caustic with E shows a lips or beak-to-beak event if $C_{4e}DA_e$ is negative or positive respectively. The paraxial eikonal Ψ, however, predicts a lips or beak-to-beak event if C_{4p} is negative or positive respectively, provided $a \neq 0$. Thus, if $a \neq 0$ and $D\kappa_2 < -4Z_0^2a^2$ then the exact and paraxial eikonals predict the opposite events! If $a = 0$ then $Y'' = Y'$, $Z'' = Z'$ ($\alpha = 0$) and the paraxial eikonal predicts a degenerate intersection (third-order contact) of the rib with the Y axis, while the exact eikonal gives a lips or beak-to-beak according as $C_{4e}D\kappa_2$ is negative or positive respectively. The case $a = 0$ is probably the more important in practice because it corresponds to observations in (X, Y) planes, which are perpendicular to the local optical axis and hence correspond to a focusing sequence.

10. Conclusions

The paraxial eikonal is an approximation derived from the exact eikonal by simply truncating its Taylor series—a technique that is common in physics, and appears justified from purely numerical considerations. The paraxial eikonal is exact in the far-field limit; otherwise it will always introduce quantitative errors, but these are expected. What is not expected, because it does not result from any violation of the purely numerical conditions imposed in the derivation, is that the paraxial eikonal may introduce serious qualitative errors. The essential point of our analysis is the warning that singularities are sensitive to approximations in subtle ways, and in general a full singularity theory analysis is essential—rule-of-thumb approximations, even if supported by physical and numerical considerations, may not work.

In all but the simplest cases of the fold (A_2) and the generic umbilics ($D_4^{\pm}$) the paraxial eikonal may, and in general will, predict the wrong type of focus on the local optical axis. Under certain conditions (which have not yet been determined) it may, however, predict the correct focus, but displaced from where it should be on the axis. Such a prediction is inconsistent with the laws of geometrical propagation. This is connected with the fact that any ray derived from the paraxial eikonal other than that along the local optical axis is not in general orthogonal to the initial wavefront. Even when it predicts the right type of focus, the paraxial eikonal cannot be relied upon to describe the caustic geometry beyond linear variations, because as our last example showed it can give the wrong sign for the caustic curvature.

These failures of the paraxial eikonal do not imply any failure of paraxial optics *per se*—they merely point out the need to make approximations with great care, and probably at a later stage than that which produces the paraxial eikonal. To analyse paraxially undetermined foci in their correct locations this is essential. It will undoubtedly complicate the algebra, but this could be alleviated by performing the algebraic manipulations by computer (e.g. see appendix 1 of Poston and Stewart (1978)).

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References

Arnol'd V I 1975 *Usp. Mat. Nauk* **30:5** 3–65 (*Transl. Russian Math. Surveys 30:5 1–75*) (Also published 1981 in *Singularity Theory: Selected Papers, London Math. Soc. Lecture Note Series* **53** (Cambridge: CUP) pp 132–206)

Berry M V, Nye J F and Wright F J 1979 *Phil. Trans. R. Soc.* A **291** 453–84

Berry M V and Upstill C 1980 in *Prog. in Optics*, vol 18, ed E Wolf (Amsterdam: North-Holland) pp 257–346

Born M and Wolf E 1975 *Principles of Optics* 5th edn (Oxford: Pergamon)

Nye J F 1978 *Proc. R. Soc.* A **361** 21–41

—— 1979 *Phil. Trans. R. Soc.* A **292** 25–44

Poston T and Stewart I N 1978 *Catastrophe Theory and its Applications* (London: Pitman)

Upstill C, Wright F J, Hajnal J V and Templer R H 1982 *Opt. Acta* **29** 1651–76

Zeeman E C 1982 *Contemporary Math.* **9** 207–72